Solution by G. W. GREENWOOD, M. A., McKendree College, Lebanon, Ill.

$$f(x)\equiv(x-1)^4(ax^3+bx^2+cx+d)-1\equiv(x+1)^4(a'x^3+b'x^2+c'x+d')+1.$$

Putting $-x$ for x and transposing we have

$$(x-1)^4(a'x^3-b'x^2+c'x-d')-1\equiv(x+1)^4(ax^3-bx^2+cx-d)+1.$$

These identities are consistent if $a=a'$, $b=-b'$, $c=c'$, $d=-d'$.

By equating coefficients of the same powers of x in either identity, and solving the resulting equations we get $a=\frac{5}{16}$, $b=\frac{5}{4}$, $c=\frac{29}{16}$, $d=1$.

$\therefore f(x)\equiv\frac{5}{16}x^7-\frac{21}{16}x^5+\frac{35}{16}x^3-\frac{35}{16}x.$

Also solved by R. D. Carmichael, A. H. Holmes, Henry Heaton, J. Scheffer, G. B. M. Zerr, and the Proposer.

211. Proposed by R. D. CARMICHAEL, Hartselle, Ala.

*If $x=v^{1/(v-1)}$, what is the $f(x)$ such that $v=f(x)$?

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let $v=u+1$. Then $x^u=u+1$. Let $x=e^c$. $\therefore e^{cu}=u+1$. Let $cu=y$. $\therefore e^y=y/c+1$. $\therefore y=-c+ce^y$. If $y=a+b\varphi(y)$ we have by Lagrange's Theorem

$$y=a+b\varphi(a)+\frac{b^2}{2!}\cdot\frac{d}{da}[\varphi(a)]^2+\dots\dots+\frac{b^n}{n!}\cdot\frac{d^{n-1}}{da^{n-1}}[\varphi(a)]^n+\dots\dots \text{ etc.}$$

In this problem $\varphi(a)=e^a$.

$$\therefore y=-c+ce^{-c}+c^2e^{-2c}+\frac{c^3}{2!}e^{-3c}+\frac{c^4}{3!}e^{-4c}+\dots\dots+\text{ etc.}$$

$$\therefore u+1=v=f(x)=e^{-c}+ce^{-2c}+\frac{c^2}{2!}e^{-3c}+\frac{c^3}{3!}e^{-4c}+\dots\dots \qquad c=\log_e x,\ e^{-c}=1/x.$$

$$\therefore v=f(x)=1/x+\frac{\log_e x}{x^2}+\frac{(\log_e x)^2}{2!x^3}+\frac{(\log_e x)^3}{3!x^4}+\dots\dots+\frac{(\log_e x)^n}{n!x^{n+1}}+\dots\dots \text{ etc.}$$

$$\therefore v=f(x)=x^{(1-x)/x}.$$

GEOMETRY.

267. Proposed by W. W. LANDIS, Dickinson College, Carlisle, Pa.

Prove that every orthogonal system of circles in a plane is an isothermal system.

*This problem should admit of interesting generalizations, say for $x=v^{f(v)}$ for certain classes of functions f. G.

Solution by F. H. SAFFORD, Ph. D., The University of Pennsylvania.

Let any given orthogonal system of circles be inverted with respect to a circle whose center is an intersection of two circles, one from each family, and neither of them a real circle. The result is a new orthogonal system containing two straight lines derived from the two circles, and each line is the locus of centers of the 'opposite' family of circles. Using these lines as axes of coördinates, the two circle families are $(x-a)^2+y^2=c^2$, $x^2+(y-b)^2=d^2$, in which, because the circles are orthogonal, $a^2+b^2=c^2+d^2$. Writing $c^2=a^2-k^2$ in the last equation gives $d^2=b^2+k^2$, and the circle families become $x^2+y^2-2ax+k^2=0$, $x^2+y^2-2by-k^2=0$.

The constants are now independent, but since any circle of one family is orthogonal to all of the other family it follows that a and b are the respective parameters. If now a and b are replaced by $k\coth 2v$ and $-k\cot 2u$, respectively, the system may be written $u+vi=\tan^{-1}\dfrac{x+yi}{k}$, which shows that it is isothermal. Thus the given system is also isothermal, since it may be obtained from this one by inversion. When k is 0 or ∞ the corresponding result is

$$u+vi=\frac{1}{x+yi}, \text{ or } u+vi=x+yi.$$

274. Proposed by R. D. CARMICHAEL, Hartselle, Ala.

If a straight line AB is placed between two intersecting straight lines MN and PQ and is made to revolve through all possible positions having A always in MN and B always in PQ, what is the locus of any point L in AB or AB produced?

I. Solution by G. W. GREENWOOD, M. A., McKendree College, Lebanon, Ill.

We can choose coördinate axes so that the equations to the given lines are $y=rx$, $z=a$; $y=-rx$, $z=-a$. Let the coördinates of A, B, L be, respectively, (h, rh, a), $(k, -rk, -a)$, (x, y, z). Then

$$\frac{x-h}{x-k}=\frac{y-rh}{y+rk}=\frac{z-a}{z+a}=\frac{AL}{BL}=m, \text{ say.}$$

$\therefore h-mk=x(1-m)$..............(1), $r(h+mk)=y(1-m)$...............(2), $z(1-m)=a(1+m)$...............(3).

Hence the locus lies in a plane parallel to $z=0$, or to the given lines as is otherwise evident. Also $AB^2=l^2=(h-k)^2+r^2(h+k)^2+4a^2$...............(4). Eliminating h, k between (1), (2), (4) we have an ellipse for the required locus, its equation being

$$(1-m)^2\{[y(m-1)+rx(m+1)]^2+[y(m+1)+rx(m-1)]^2r^2\}=4m^2r^2(l^2-4a^2).$$